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Exact Linear Renormalization of a One-Dimensional System with Phase Transition

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We study by real-space renormalization a class of one-dimensional self-avoiding walks (SAWs) exhibiting a nonzero critical temperature. A linear renormalization transformation is carried out in closed form in a three-parameter subspace of SAW Hamiltonians. We find lines of fixed points along which the degree of localization of the fixed-point interactions varies. The role of the spin rescaling factor in the transformation is explicitly demonstrated.

KEY WORDS: Real-space renormalization; one-dimensional model; self-avoiding walk; linear renormalization group; spin rescaling; lines of fixed points.

1. INTRODUCTION

In this paper we carry out a case study in renormalization group theory, employing the real-space renormalization method of Niemeyer and van Leeuwen[1,2] (NvL). The case at hand is a certain class of one-dimensional ($d = 1$), n-component spin systems, considered in the limit $n \to 0$. A closely related $d = 1$ system was first studied by Balian and Toulouse.[3] These authors showed that if the number of spin components becomes less than 1 ($n = 1$ corresponds to the Ising case), there occurs a phase transition at a finite temperature. The associated critical exponents are nonclassical integers, viz. $\alpha = \gamma = \eta = \nu = 1$ and $\beta = 1/\delta = 0$. Since the work of de Gennes,[4] des Cloizeaux,[5] and others[6–9] it has been known that the $n \to 0$ limit of an n-component spin system on a general lattice yields the description of a self avoiding walk (SAW) on that lattice. This correspondence—formulated in a particularly clear and concise way in the appendix of Ref. 9—has been fruitful both theoretically and experimentally[9] in polymer research.

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It was recently shown[10,11] how the NvL renormalization method can be used in connection with the de Gennes–des Cloizeaux equivalence to obtain approximate values for the critical exponents of the self-avoiding walk problem. The present note is an application to the one-dimensional case of the technique developed in Ref. 11. The SAW problem on a one-dimensional lattice is of course trivial. We shall see that it is nevertheless worthwhile to study this model from the point of view of renormalization group theory, since one is able to derive rigorous results which clarify the working of a real-space renormalization group.

Of the existing literature on exactly renormalizable systems, a large part is concerned with one-dimensional models. Especially worth mentioning is a thorough discussion by Nelson and Fisher[12] of a variety of one-dimensional Ising systems, in which many of the features of renormalization group theory are illustrated by examples. Krinsky and Furman[13] consider an exactly renormalizable one-dimensional spin-1 Ising model, Bell and Wilson[14] the Gaussian model in arbitrary dimension, Bleher and Sinai[15] certain modifications of Dyson's hierarchical models, and Niemeyer and Ruijgrok[16] the classical Heisenberg chain.

In this work we shall employ a linear renormalization transformation depending on a spin rescaling factor q. It will prove possible to find closed-form renormalization equations in a three-parameter subspace of SAW Hamiltonians. We shall find that there are lines of fixed points along which the degree of localization of the fixed-point interactions varies. The role of the spin rescaling factor in the transformation will be explicitly demonstrated and discussed, and the connection with existing work pointed out.

2. THE MODEL

We consider a one-dimensional lattice with sites $i = 1, 2, \ldots, N$ and periodic boundary conditions. At each site there will be an n-component spin $\boldsymbol{\sigma}_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in})$ of length n, which takes only the $2n$ discrete values

$$\boldsymbol{\sigma}_i = (0,\ldots,0, \pm n^{1/2}, 0,\ldots, 0) \tag{1}$$

These spin variables, in contrast to the spherical spins employed by Balian and Toulouse, have cubic symmetry. However, it will be seen that this difference does not change the nature of the phase transition. We briefly list some important properties of the spins in Eq. (1), which are easily verified:

$$\begin{gathered} \sum_{\alpha=1}^{n} \sigma_{i\alpha}^2 = n, \qquad \sigma_{i\alpha}\sigma_{i\beta} = \sigma_{i\alpha}^2 \delta_{\alpha\beta}, \qquad \sigma_{i\alpha}^p = n\sigma_{i\alpha}^{p-2} \quad (p \geqslant 3) \\ (2n)^{-1} \sum_{\boldsymbol{\sigma}_i} \sigma_{i\alpha}^2 = 1, \qquad \lim_{n\to 0} (2n)^{-1} \sum_{\boldsymbol{\sigma}_i} \sigma_{i\alpha}^p = \delta_{p0} + \delta_{p2} \end{gathered} \tag{2}$$

In Ref. 11 it was shown how one can describe the SAW problem on an arbitrary lattice by a Hamiltonian expressed in spins of type (1). In the present one-dimensional case we shall be interested in spin operators A_{ij} defined by

$$A_{ij} = \sum_{\alpha} \sigma_{i\alpha}\sigma_{i+1,\alpha}^2 \cdots \sigma_{j-1,\alpha}^2\sigma_{j\alpha}, \qquad j > i \tag{3}$$

According to the interpretation of Ref. 11, such an operator corresponds to a segment of a self-avoiding walk between the *end sites* i and j, and visiting all *intermediate sites* $i + 1$, $i + 2$,..., $j - 1$. We shall also need to consider the operator A_{ii} defined by

$$A_{ii} = \sum_{\alpha} \sigma_{i\alpha}^2\sigma_{i+1,\alpha}^2 = \left(\sum_{\alpha} \sigma_{i\alpha}\sigma_{i+1,\alpha}\right)^2 \tag{4}$$

This operator clearly corresponds to a closed loop from site i to site $i + 1$ and back. In the sequel we shall employ the terms *segment operator* and *closed loop operator* for expressions of the form of Eqs. (3) and (4), respectively. Two operators will be called *disconnected* if they do not involve any common sites, *adjacent* if they involve only common end sites, and *overlapping* otherwise.

Let us consider the (reduced) Hamiltonian

$$H = -J_0 \sum_{i=1}^{N} A_{ii} - J \sum_{i=1}^{N} \sum_{k=1}^{N-1} \mu^{k-1} A_{i,i+k} \tag{5}$$

The interaction constants J_0 and $J\mu^{k-1}$ will be referred to as the *weights* of the corresponding operators. The partition function Z_n is defined by

$$Z_n = (2n)^{-N} \sum_{\{\boldsymbol{\sigma}_i\}} e^{-H} \tag{6}$$

By choosing the normalization factor $(2n)^{-N}$, we have added to the free energy a singular term $N \log 2n$, which is just the entropy of N free n-component spins. The factor $(2n)^{-N}$ of course drops out in the calculation of averages with respect to e^{-H}. As we shall see, for $n \to 0$ the partition function (6) differs from unity only by terms of order n. We therefore define the free energy per spin and *per spin component* as

$$f = -\lim_{N\to\infty} \lim_{n\to 0} (Nn)^{-1} \log Z_n \tag{7}$$

Note that we let $n \to 0$ first, so that we are actually taking the thermodynamic limit of a system of zero-component spins.

The $n = 0$ average of an arbitrary function A of the spin variables is defined by

$$\langle A \rangle = \lim_{N\to\infty} \lim_{n\to\infty} \langle A \rangle_n, \qquad \langle A \rangle_n = Z_n^{-1}(2n)^{-N} \sum_{\{\boldsymbol{\sigma}_i\}} A\{\boldsymbol{\sigma}_i\} e^{-H} \tag{8}$$

Special averages of interest will be the pair correlations obtained by taking $A\{\boldsymbol{\sigma}_i\} = \sigma_{j1}\sigma_{k1}$. We remark that general correlation functions can also be expressed as field derivatives of $f\{H_i\}$, a quantity defined as the free energy of the Hamiltonian (5) to which a term $\sum_i \sum_\alpha H_{i\alpha}\sigma_{i\alpha}$ has been added.

Before passing on to renormalization, we shall investigate some properties of this model by classical methods. The exponential in Eq. (6) is the generating function for self-avoiding walks. By expanding it in a Taylor series, one can "hook" the elementary walk segments A_{ij} together to form longer walks. Indeed, using (3) and the second of relations (2), we have

$$A_{ij}A_{jl} = A_{il} \tag{9}$$

With the aid of relations (2) the sum on $\{\boldsymbol{\sigma}_i\}$ in (6) can be calculated for each term in the Taylor expansion. Whenever in the expansion a term occurs with a product of overlapping segments, the third of relations (2) introduces extra factors n. The contribution of such a term to the partition function Z_n is of order n^2, and its contributions to the free energy and the correlation functions vanish proportionally to n as $n \to 0$. In general we shall call a spin operator A constructed from the A_{ij} *of order* n^p if products containing A contribute to Z_n only to order n^p and higher. One can establish the following properties, which we number for easy reference:

(P1) The segment operators A_{ij} are of order n.
(P2) A product of adjacent or disconnected segment operators is of order n.
(P3) A product of overlapping operators is of order n^2 or higher.
(P4) The product of a closed-loop operator and any other operator is of order n^2 or higher.

Each term in the Taylor expansion of e^{-H} naturally corresponds to a lattice graph. For a general lattice one can show without difficulty that the free energy f is a sum of contributions stemming from all the closed non-intersecting loops that can be constructed on the lattice. Products of disconnected loops are by (P4) of order n^2 and do not contribute as $n \to 0$. In the present case the only possible loops are (i) those starting from a site i, visiting site $i + 1$, and returning to i, and (ii) a single loop of length N all around the chain. The type (i) loops arise only in the first and second terms of the Taylor expansion and their weights are easily found in terms of J_0, J, and μ. Let the large loop consist of L segments of lengths $k_1, k_2, \ldots, k_L$. Denoting its contribution for $n \to 0$ by C_N, we have

$$f = -J_0 - \tfrac{1}{2}J^2 - \lim_{N\to\infty} N^{-1}C_N \tag{10}$$

with

$$C_N = \sum_{L \geqslant 1} \sum_{k_1 \geqslant 1} \cdots \sum_{k_L \geqslant 1} \delta(k_1 + \cdots + k_L - N) J^L \mu^{\Sigma_{l=1}^{L}(k_l - 1)}$$
$$= (\mu + J)^N - \mu^N \tag{11}$$

The presence of the last term in Eq. (10) is evidently due to the choice of cyclic boundary conditions; in the case of free boundaries it would be absent. Thus no thermodynamic limit exists unless $|\mu| \leqslant 1$ and $|\mu + J| \leqslant 1$.

The pair correlation $\langle \sigma_{01} \sigma_{j1} \rangle$ can similarly be shown[11] to be a sum of contributions, each corresponding to a self-avoiding walk beginning at the origin and ending at site j. By summing as in (11) on all possible linear sequences of adjacent walk segments, one finds

$$\langle \sigma_{01} \sigma_{j1} \rangle = J(\mu + J)^{j-1} = J(\mu + J)^{-1} e^{-j/\xi_{\text{corr}}} \tag{12}$$

where the correlation length ξ_{corr} is given by

$$\xi_{\text{corr}} = [-\log(\mu + J)]^{-1} \tag{13}$$

Using Eq. (12) and the property that $\langle \sigma_{01}^2 \rangle = 1$, we find for the susceptibility

$$\chi \equiv \sum_{j=-\infty}^{\infty} \langle \sigma_{01}, \sigma_{|j|1} \rangle = \frac{(1 - \mu)^2 - J^2}{(1 - \mu - J)^2} \tag{14}$$

This shows that χ diverges for $\mu + J = 1$, which equation indicates a hyperplane of critical Hamiltonians. For the critical exponents η, ν, and γ, defined in the usual way, we easily find from Eqs. (12)–(14)

$$\eta = 1, \qquad \nu = 1, \qquad \gamma = 1 \tag{15}$$

independently of μ and J. For the special case with $J_0 = 0$ and $\mu = 0$ (so that there is only a nearest neighbor coupling J) it is possible to calculate Z_n and the correlation functions explicitly for arbitrary n, e.g., by the transfer-matrix method. One finds

$$Z_n = n\left(\frac{\sinh nJ}{n}\right)^N + (n - 1)\left(\frac{\cosh nJ - 1}{n}\right)^N + \left(\frac{\cosh nJ + n - 1}{n}\right)^N \tag{16}$$

From this expression one easily deduces the free energy f according to Eq. (7) and finds a result in agreement with (10). The order of the two limits in (7) appears to be unimportant. This allows us to assign a value to the exponent α in the following way. At fixed $n > 0$ and for $N \to \infty$ we may obtain from (16) a free energy f_n given by

$$f_n = -\frac{1}{n} \max\left(\log \frac{\sinh nJ}{n}, \log \frac{\cosh nJ + n - 1}{n}\right) \tag{17}$$

leading to a transition with $\alpha = 1$. Thus we have by analytic continuation

that α equals 1 also for $n = 0$. We remark finally that the exponents γ, α, ν, and η found above for the case of a short-range interaction are identical to those found by Balian and Toulouse[3] and satisfy the usual scaling relations.

3. THE RENORMALIZATION TRANSFORMATION

In this section we shall define a renormalization transformation for the Hamiltonian (5). A more general discussion of such transformations for $n = 0$ Hamiltonians can be found in Ref. 11. We divide the lattice up into cells of two spins each and associate with every pair $\boldsymbol{\sigma}_{2i-1}$ and $\boldsymbol{\sigma}_{2i}$ a cell spin $\boldsymbol{\mu}_i$. A renormalization transformation is defined[2] by the matrix

$$P(\{\boldsymbol{\mu}_j\}, \{\boldsymbol{\sigma}_i\}) = \prod_j p(\boldsymbol{\mu}_j; \boldsymbol{\sigma}_{2j-1}, \boldsymbol{\sigma}_{2j}) \tag{18}$$

with the weight factor p normalized such that

$$(2n)^{-1} \sum_{\boldsymbol{\mu}} p(\boldsymbol{\mu}; \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = 1 \tag{19}$$

The renormalized Hamiltonian H' corresponding to H is then defined by

$$\begin{aligned} \exp(-nG - H'\{\boldsymbol{\mu}_j\}) &= (2n)^{-N} \sum_{\{\boldsymbol{\sigma}_i\}} P(\{\boldsymbol{\mu}_j\}, \{\boldsymbol{\sigma}_i\}) \exp(-H\{\boldsymbol{\sigma}_i\}) \\ &\equiv R[\exp(-H)] \end{aligned} \tag{20}$$

where G is chosen such that $H'\{\boldsymbol{\mu}_j\}$ contains no constant (spin-independent) term. More in general we define for any spin operator $A\{\boldsymbol{\sigma}_i\}$ its renormalized counterpart

$$R[A] = (2n)^{-N} \sum_{\{\boldsymbol{\sigma}_i\}} P(\{\boldsymbol{\mu}_j\}, \{\boldsymbol{\sigma}_i\}) A\{\boldsymbol{\sigma}_i\} \tag{21}$$

In this work we shall employ a one-parameter renormalization transformation R_q specified by the following choice[11] for p:

$$p(\boldsymbol{\mu}; \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = \tfrac{1}{4} \sum_{\alpha} [n^{-2} \mu_\alpha{}^2 (\sigma_{1\alpha}^2 + \sigma_{2\alpha}^2) + n^{-1} q \mu_\alpha (\sigma_{1\alpha} + \sigma_{2\alpha})] \tag{22}$$

The transformation R_q is entirely determined by the single-cell averages of an arbitrary cell i. For these one obtains with the help of (21) and (22)

$$R_q[1] = 1 \tag{23a}$$

$$R_q[\sigma_{2i,\gamma}^2] = \tfrac{1}{2}(1 + \mu_{i\gamma}^2) \tag{23b}$$

$$R_q[\sigma_{2i-1,\gamma_1}^2 \sigma_{2i,\gamma_2}^2] = \tfrac{1}{2}(\mu_{i\gamma_1}^2 + \mu_{i\gamma_2}^2) \tag{23c}$$

$$R_q[\sigma_{2i-1,\gamma_1} \sigma_{2i,\gamma_2}] = 0 \tag{23d}$$

$$R_q[\sigma_{2i,\gamma}] = \tfrac{1}{2} q \mu_{i\gamma} \tag{23e}$$

$$R_q[\sigma_{2i-1,\gamma_1} \sigma_{2i,\gamma_2}^2] = \tfrac{1}{2} q \mu_{i\gamma_1} \tag{23f}$$

With the aid of the symmetry properties of $p(\boldsymbol{\mu}; \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ one easily finds the remaining cell averages.

An important characteristic of R_q is its linearity. Properties of linear transformations have been discussed by Bell and Wilson,[14,17] Nelson and Fisher,[12] and Niemeyer and van Leeuwen.[18],[2] For our case the main results are easily rederived as follows. Let $\langle \cdots \rangle'$ denote the thermal average with respect to the renormalized Hamiltonian $H'\{\boldsymbol{\mu}_i\}$. Let furthermore $\mathbf{J}'$ stand for a set of renormalized coupling constants obtained from an initial set $\mathbf{J}$. Reexpressing the pair correlation function $G(\mathbf{J}', r) \equiv \langle \mu_{0\gamma} \mu_{r\gamma} \rangle'$ in terms of the original site spins, one finds

$$G(\mathbf{J}', r) = \tfrac{1}{4} q^2 [G(\mathbf{J}, 2r - 1) + 2G(\mathbf{J}, 2r) + G(\mathbf{J}, 2r + 1)] \tag{24}$$

whence for $r \to \infty$

$$G(\mathbf{J}', r) \approx q^2 G(\mathbf{J}, 2r) \tag{25}$$

It is because of this proportionality between the pair correlations that one calls R_q a linear transformation.[3] A special property of such a linear transformation is derived in the usual way.[17-22] Suppose that the interactions $\mathbf{J}, \mathbf{J}', \ldots$ converge to a fixed-point interaction $\mathbf{J}^*$ and that $G(\mathbf{J}^*, r) \approx A r^{1-\eta}$ as $r \to \infty$. Substitution in (25) yields the well-known relation

$$q = q^* \equiv 2^{(\eta - 1)/2} \tag{26}$$

i.e., a connection between the seemingly arbitrary parameter q in the transformation and the physical exponent η. We have to conclude that unless q takes the special (but unknown) value q^* given by (26), the transformation cannot have a fixed point with a correlation function as assumed. In approximate calculations with linear transformations the proper adjustment of the parameter presents a considerable problem, since one usually finds fixed points in a whole range of parameter values. This difficulty is extensively discussed by Bell and Wilson.[14] The example of the next section will show explicitly what happens to the transformation for improper values of q.

General considerations allow a further prediction. Following Bell and Wilson[14] and Niemeyer and van Leeuwen,[18] one may define a Legendre transformation L_s (which does not reduce the number of spins) by the weight factor

$$p(\boldsymbol{\mu}_i, \boldsymbol{\sigma}_i) = \tfrac{1}{2} \sum_{\alpha} (n^{-2} \mu_{i\alpha}^2 \sigma_{i\alpha}^2 + n^{-1} s \mu_{i\alpha} \sigma_{i\alpha}) \tag{27}$$

[2] Nonlinear renormalization transformations have been studied by Bell and Wilson[17] by perturbation theory. The best known nonlinear transformation is perhaps the so-called sign-rule transformation.[1] For an exactly soluble nonlinear case see Nelson and Fisher.[12]

[3] The name is misleading: The transformation $\mathbf{J} \to \mathbf{J}'$ is not linear. Niemeyer and van Leeuwen[2] prefer to speak instead of "linear weight factors," but this term is not quite appropriate here either in view of the quadratic terms in (22).

One shows easily that L_s and R_q commute. It follows in the usual way that if R_{q^*} has a fixed-point Hamiltonian H^*, then it has a one-parameter family of fixed-point Hamiltonians H_s^* given by

$$\exp(-nG_s - H_s^*) = L_s[\exp(-H^*)] \tag{28}$$

Since L_s leaves the partition function invariant, all Hamiltonians H_s^* obtained from a given H^* have the same critical properties. Moreover, one expects the transformation R_q to possess a marginal eigenvalue $\lambda_1 = 1$ along the line H_s^*.[4] All these properties will also be illustrated explicitly in the next section.

4. AN EXACTLY RENORMALIZABLE CASE

In Appendix A it is shown that if one applies to the Hamiltonian H given in (5) the renormalization transformation R_q and expands in powers of n, then the new Hamiltonian H' is, to lowest order in n, again of the form (5), but with renormalized coupling constants. The detailed calculations show that for $N \to \infty$ we have the following explicit recursion:

$$\mathbf{J}' = \tfrac{1}{4}q^2 J(1 + \mu + J)^2 \tag{29a}$$

$$\mu' = (\mu + J)^2 - \tfrac{1}{4}q^2 J(1 + \mu + J)^2 \tag{29b}$$

Furthermore, the renormalized coupling J_0' and the constant G are related to the original couplings by

$$J_0' = \tfrac{1}{4}(J_0 + \tfrac{1}{2}J^2) - \tfrac{1}{32}q^4 J^2(1 + \mu + J)^4 \tag{30}$$

$$G = \tfrac{7}{8}N(J_0 + \tfrac{1}{2}J^2) \tag{31}$$

Hence we have found a three-parameter family of interactions, depending on μ, J, and J_0, which is invariant under the one-parameter transformation R_q. The renormalization equations (29)–(31) depend only on q^2, so that we may take $q \geqslant 0$. They assume a somewhat more elegant form in the variables J, $\xi \equiv \mu + J$, and $J_0 + \frac{1}{2}J^2$. One finds that the ensemble of fixed points has the following structure, (i) $q^* = 1$, $\xi^* = 1$, J arbitrary; (ii) $q^* = 2$, $\xi^* = 0$, J arbitrary; (iii) q and ξ arbitrary, $J^* = 0$. With the aid of (30) one can also find the corresponding fixed-point values J_0^* in the three cases. The fixed points of type (iii) are trivial and do not need further discussion. Those of type (i) and those of type (ii) form lines of fixed points parametrized by the coupling strength J. The corresponding fixed-point values of the interactions $J\mu^{k-1}$ in the Hamiltonian (5) are:

$$q^* = 1: \qquad (J\mu^{k-1})^* = J(1 - J)^{k-1} \tag{32}$$

$$q^* = 2: \qquad (J\mu^{k-1})^* = J(-J)^{k-1} \tag{33}$$

[4] For a more general discussion of the invariance properties of the renormalization group see Wegner.[23]

We consider first the case $q^* = 1$. In this case the effective range of the fixed-point interaction is $|\log|1 - J||^{-1}$. For $J = 1$ the fixed-point Hamiltonian is a pure nearest neighbor one. As $|1 - J|$ grows, the fixed-point interaction becomes less and less localized, until for $|1 - J| > 1$ it becomes unphysical. The fixed-point matrix $[\partial(\mu', J', J_0')/\partial(\mu, J, J_0)]^*$ is easily calculated and diagonalized. One finds for the three eigenvalues λ_T, λ_1, and λ_2, and for the critical exponent $\nu = \log 2/\log \lambda_T$,

$$\lambda_T = 2, \qquad \lambda_1 = 1, \qquad \lambda_2 = \tfrac{1}{4}; \qquad \nu = 1 \tag{34}$$

These results are independent of the value of J, as they should be. Thus, as expected, we find one relevant and one marginal eigenvalue. By setting $q = 1$ in Eq. (26), one obtains, furthermore, $\eta = 1$. Hence the values for ν and η agree with those found in Section 2 from the direct solution.

The line of fixed points obtained for $q^* = 2$ passes through $J = \mu = 0$ and hence is a line of infinite-temperature fixed points. For $|J| > 1$ the interaction becomes unphysical. The fixed-point matrix has again a marginal eigenvalue 1, and furthermore the eigenvalues $\frac{1}{4}$ and 0.

We shall now study the flow diagram in the $J\xi$ plane that arises if one iterates the transformation R_q. The third coordinate J_0 will be disregarded. For the $(l + 1)$th iteration, Eqs. (30) read in terms of J and ξ

$$J_{(l+1)} = \tfrac{1}{4}q^2 J_{(l)}(1 + \xi_{(l)})^2 \tag{35a}$$

$$\xi_{(l+1)} = \xi_{(l)}^2 \tag{35b}$$

After the first iteration ξ has a nonnegative value. The transformation (35) is furthermore symmetric with respect to $J \to -J$. We can therefore restrict ourselves to the positive $J\xi$ quadrant. From initial values $\xi = \xi_{(0)}$ and $J = J_{(0)}$ one obtains after l iterations

$$\xi_{(l)} = \xi_{(0)}^{2^l} \tag{36a}$$

$$J_{(l)} = (\tfrac{1}{4}q^2)^l \left(\frac{1 - \xi_{(0)}^{2^l}}{1 - \xi_{(0)}}\right)^2 J_{(0)} \tag{36b}$$

A flow field is obtained as $[d(J_{(l)}, \xi_{(l)})/dl]_{l=0}$. Qualitative flow diagrams have been plotted for five different cases: $0 < q < 1$; $q = 1$; $1 < q < 2$; $q = 2$; and $q > 2$ (Fig. 1). The line $\xi = 1$ represents the only nontrivial set of fixed points. It is a fixed line only when $q = 1$. The flow diagrams show what happens if $q \neq 1$. The line $\xi = 1$ of critical Hamiltonians is still invariant. This should of course be so, since the transformation R_q, even with the "wrong" q, leaves the partition function invariant, due to Eq. (19). However, for $q \neq 1$, the transformation R_q moves the initial Hamiltonian along the critical line to an unphysical part of that line. For $q < 1$, the interaction strength J tends to zero and the range of the fixed-point interaction tends to

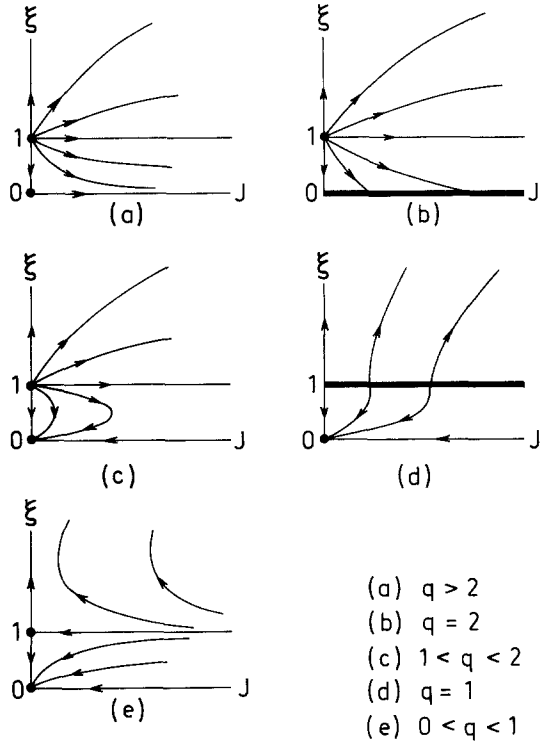

Fig. 1. Qualitative flow diagrams in the $J\xi$ plane resulting from Eqs. (35). Dots are fixed points; heavy lines are fixed lines.

infinity. For $q > 1$, the interaction strength J tends to infinity, and the fixed-point interactions $(J\mu^{k-1})^*$ become infinitely strong but oscillatory in the distance k.

This shows explicitly that proper adjustment of the parameter q is needed to keep the strength and the range of the interactions in check in the course of the iterations. The parallel with Wilson's ϵ expansion[19–22] of the Ginzburg–Landau Hamiltonian is obvious. In this Hamiltonian the interaction is represented by a squared gradient term of the spin field. This term is by its nature always short-ranged, but its strength may vary. In the ϵ expansion an undetermined spin rescaling factor occurs (usually called ζ), which is adjusted such that the strength of the gradient term remains equal to a fixed constant. Only then can one find a fixed point, and a relation between ζ and η similar to Eq. (26) holds. Thus our parameter q may rightfully be considered as the spin rescaling factor of the real-space renormalization method.

5. CONCLUSION

A generalized one-dimensional self-avoiding walk, exhibiting a nonzero critical temperature, offers an instructive example of the working of a linear renormalization group. A one-parameter renormalization transformation has been carried out in closed form in a subspace of self-avoiding-walk Hamiltonians. Lines of fixed points were obtained along which the degree of localization of the interactions varies. Flow fields in Hamiltonian space have been constructed for different values of the parameter q in the transformation. The close analogy of q to the spin rescaling factor in the ϵ expansion has been pointed out.

APPENDIX A. DERIVATION OF THE RECURRENCE RELATIONS FOR H' AND G

In this appendix we shall calculate in the limit $n \to 0$ the renormalized Hamiltonian H' and the free energy contribution G. Both are formally defined by Eq. (20), in which we take for R the transformation R_q defined by (21), (18), and (22) and for the initial Hamiltonian H the expression (5).

Before embarking upon the actual calculation we shall discuss a number of simplifications that occur in the renormalization procedure as one lets n tend to zero. In this limit one can neglect in the renormalized expression $nG + H'$ any terms that are of order n^2 or higher, i.e., whose contributions to the free energy (7) and to the correlation functions (12) vanish. The order in n of any spin operator occurring in the calculation is easily determined according to the rules (P1)–(P4) of Section 2. With the aid of the renormalization rules (23) one also verifies that the transformation R_q leaves the order in n of an operator unchanged.

We shall call a general product $A = A_{i_1 j_1}^{a_1} A_{i_2 j_2}^{a_2} \cdots$ (with all $a_k \geqslant 1$) *cell-disconnected* if the operators $A_{i_k j_k}$ can be divided into two groups not involving any common cell. Since R_q factorizes over the cells, the average $R_q[A]$ is just the product of the averages of the cell-connected groups in A. These averages can be calculated by Eqs. (23). It is not hard to verify that the only cell-connected groups whose averages neither vanish nor are of order n^2 belong to one of the three following types:

$$A_{ii} \tag{A1a}$$

$$A_{i,i+1}^2 = A_{ii} \tag{A1b}$$

$$A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{k-1} i_k} = A_{i_1 i_k} \tag{A1c}$$

The averages of the operator products in (A1) are easily calculated with the aid of (23). Let for given i and k the integers I and K be such that $i = 2I - 1$

or $i = 2I$, and $k = 2K - 1$ or $k = 2K$. Let the operators A'_{IK} be defined in a manner analogous to (3), but with μ spins instead of σ spins. Then we find

$$R_q[A_{ii}] = n\delta_{i,2I-1} + \tfrac{3}{4}n\delta_{i,2I} + \tfrac{1}{4}\delta_{i,2I}A'_{II} \tag{A2a}$$

$$R_q[A_{2I-1,2I}] = 0 \tag{A2b}$$

$$R_q[A_{ik}] = \tfrac{1}{4}q^2 A'_{IK}, \qquad k > i, \quad K > I \tag{A2c}$$

Relation (A2c) expresses that a walk segment between two sites i and k is mapped onto a walk segment between the corresponding cells I and K. Intermediate sites of the walk become intermediate cells. Relations (A2a) and (A2b) refer to special cases. In the sequel we shall show how one can find the renormalized $nG + H'$ with the aid of (A2) and the rules (P1)–(P4) concerning the order in n.

We recall that R_q can be considered as a $\{\mu_I\}$-dependent average over the variables $\{\sigma_i\}$. Let R_q^{cum} denote the corresponding cumulant average. Thus we can write for (20) the following cumulant expansion:

$$\exp(-nG - H') = \exp \sum_{\{\nu_{ij}\}}{}' \prod_{ij} \frac{J_{j-i}^{\nu_{ij}}}{\nu_{ij}!} R_q^{\text{cum}}\left[\prod_{ij} A_{ij}^{\nu_{ij}}\right] \tag{A3}$$

where we have abbreviated $J_k \equiv J\mu^{k-1}$ for $k = 1, 2, \ldots$. The prime indicates that the summation is to be performed on all sets $\{\nu_{ij}\}$ with $\nu_{ij} > 0$, with the exception of the case where all ν_{ij} vanish. The expression for R_q^{cum} in the terms of R_q reads[24]

$$R_q^{\text{cum}}\left[\prod_{ij} A_{ij}^{\nu_{ij}}\right] = \prod_{ij} \nu_{ij}! \sum_{l=1}^{\nu} \sum_{(k_m, \mu_{mij})}{}'' (k-1)!\,(-1)^{k-1} \sum_{m=1}^{l} \frac{1}{k_m!} \left(\frac{R_q\left[\prod_{ij} A_{ij}^{\mu_{mij}}\right]}{\prod_{ij} \mu_{mij}!} \right)^{k_m} \tag{A4}$$

Each term $\{\nu_{ij}\}$ in (A4) corresponds to a decomposition of a collection of $\nu = \sum_{ij} \nu_{ij}$ objects [with ν_{ij} the number of objects of type (i, j)] into $k = \sum_{m=1}^{l} k_m$ subsets (with k_m the multiplicity of the mth subset). The double prime on the inner summation sign indicates that the set of nonnegative integers $\{k_m, \mu_{mij}\}$ has to satisfy the relation $\sum_m k_m \mu_{mij} = \nu_{ij}$. General cumulant properties ensure that the cumulant (A4) vanishes unless the product $\prod_{ij} A_{ij}^{\nu_{ij}}$ is cell-connected.

The above discussion showed that the product on l in (A4) is a product of segment operators and closed-loop operators in the μ spins, and of constants of order n. The only cases to be retained (all others being of order n^2 or higher) are (i) those where the product contains only a single closed-loop operator or constant of order n; (ii) those where the product contains an arbitrary number of nonoverlapping (open-ended) segment operators. Case

(ii) contains the special case (iia) where we have two identical segment operators $A'_{I,I+1}$ which together form a closed-loop operator. In the general case, to be denoted by (iib), one easily sees that the only contributions stem from products containing a single sequence of adjacent segment operators: by (A2), disconnected sequences can stem only from a cell-disconnected product $\prod_{ij} A_{ij}^{\nu_{ij}}$, which is known not to contribute. From these arguments we conclude that the renormalized Hamiltonian H' is again of the form (5).

We shall now calculate the new coupling constants J_k' and the free energy contribution G. According to (A4) contributions to case (i) arise in only two different types of cumulants, viz. (a) if $\nu_{ij} = \delta_{ir}\delta_{jr}$ for some r, or (b) if $\nu_{ij} = 2\delta_{ir}\delta_{j,r+1}$ for some r. For the type (a) cumulants the summations in (A4) reduce to one single term; for the type (b) cumulants Eq. (A4) contains two terms of which only one contributes to case (i). The total contribution to case (i) is collected by substituting the relevant terms from the cumulants (A4) into (A3) and summing on r. Indicating this contribution by the superscript (1), we find, using that $J_k = J\mu^{k-1}$ for $k \geqslant 1$,

$$[-nG - H']^{(1)} = J_0 \sum_r R_q[A_{rr}] + \tfrac{1}{2}J^2 \sum_r R_q[A^2_{r,r+1}]$$
$$= \tfrac{1}{2}Nn(\tfrac{7}{4}J_0 + \tfrac{7}{8}J^2) + (\tfrac{1}{4}J_0 + \tfrac{1}{8}J^2) \sum_I A'_{II} \qquad \text{(A5)}$$

Contributions to case (iia) arise only in those cumulant terms in (A4) that are the product of two R_q averages, i.e., that have either $l = 1$ and $k_1 = 2$, or $l = 2$ and $k_1 = k_2 = 1$. In case (iia) both R_q averages are equal to an operator $A'_{I,I+1}$ for an arbitrary I, and hence all contributions to this case can be found by considering all possible $\prod_{ij} A_{ij}^{\nu_{ij}}$ whose average is of type $A'_{I,I+1}$. The only A_{ij} operators involved are $A_{i,i+1}$, $A_{i,i+2}$, and $A_{i,i+3}$. Collecting all possibilities, using (A2), and summing on I, we find for the case (iia) contributions to $nG + H'$

$$[-nG - H']^{(2)} = -\tfrac{1}{32}q^4J^2(1 + \mu + J)^4 \sum_I A'_{II} \qquad \text{(A6)}$$

Combining (A5) and (A6), we obtain the expressions (30) and (31) for J_0' and G.

Finally, we shall find all contributions from the cumulants to the operators $A'_{I,I+R}$ for arbitrary $R \geqslant 1$ [case (iib) above]. The operator $A'_{I,I+R}$ can arise as a product of at most R different R_q averages. Each R_q average, in turn, is (in a unique way) a product of disconnected segments, each segment being the average of a cell-connected group. We shall first select all contributions where $A'_{I,I+R}$ is formed from s such disconnected segments ($s = 1, 2, \ldots, R$). These segments can be located on the μ lattice in $\binom{R-1}{s-1}$ different ways, $\binom{R-1}{s-1}$ being the number of ways of putting $s - 1$ "segment separations" on the intermediate sites $I + 1, \ldots, I + R - 1$. Each subset of segments not

containing any pair of adjacent segments may occur as the result of a single R_q average. Let c_{sk} be the number of ways to divide a configuration of s segments into k such subsets. By (A3) a product of k of the R_q averages has a factor $(k-1)!\,(-1)^{k-1}$. Consider first a single segment in the interval between I and $I+R$, say running from K to $K+L$. Such a segment may arise, by (A2), as the image of a segment on the σ lattice starting in cell K and ending in cell $K+L$. This gives four possible σ segments of lengths $l = 2L-1$, $2L$, $2L$, and $2L+1$. The weight of a σ segment of length l is $J^r\mu^{1-r}$, where r is the number of different operators A_{ij} that form the segment. There are $\binom{l-1}{r-1}$ ways to obtain a segment of length l from r operators A_{ij}. Hence the total weight w_l of a segment of length l is

$$w_l = \sum_{r=1}^{l} \binom{l-1}{r-1} J^r \mu^{l-r} = J(\mu + J)^{l-1} \tag{A7}$$

The weight w_L' of the image segment of length L on the μ lattice is found by combining the four possibilities and renormalizing according to (A2). One gets

$$w_L' = \frac{1}{4}q^2(w_{2L-1} + 2w_L + w_{2L+1}) = \frac{1}{4}q^2 J\left(\frac{1+\mu+J}{\mu+J}\right)^2(\mu+J)^{2L} \tag{A8}$$

The contribution $J_R(s,k)$ to the weight of $A'_{I,I+R}$ is just the product of the weights of the s segments, multiplied by the combinatorial factor c_{sk},

$$J_R(s,k) = c_{sk}\left(\frac{1}{4}q^2\right)^s\left(\frac{1+\mu+J}{\mu+J}\right)^{2s}(\mu+J)^{2R} \tag{A9}$$

The expression (29) for $J_R' \equiv J'\mu'^{R-1}$ is obtained by summing (A9) on m and s and using the identity

$$\sum_{k=1}^{s} (k-1)!\,(-1)^k c_{sk} = (-1)^s \tag{A10}$$

which is proved in Appendix B.

APPENDIX B

Let c_{sm} be the number of ways in which one can divide s linearly ordered objects ($s = 1, 2, \ldots$) into m different groups ($m = 1, 2, \ldots, s$) such that no neighboring pair is in the same group. From this definition follows the recursion relation

$$c_{sm} = (m-1)c_{s-1,m} + c_{s-1,m-1} \tag{B1}$$

Defining $c_s \equiv \sum_{m=1}^{s}(-1)^{m-1}(m-1)!\,c_{sm}$, one derives easily with the aid of (B1) the recursion $c_s = -c_{s-1}$. With the boundary condition $c_1 = c_{11} = 1$ this yields (A10).

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